On the boundary conditions at an insulating wall for hydromagnetic waves in a cylindrical plasma

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The order of the dispersion relation for the propagation of hydromagnetic waves along a magnetized cylindrical plasma falls by unity when the plasma resistivity, σ^{-1}, tends to zero. A consequence of this is that the two boundary conditions necessary on an insulating wall are reduced to a single condition, a reduction brought about by the development of a current sheet. If the ratio, $\Omega \equiv \omega/\omega_{ci}$, of the wave frequency to the ion cyclotron frequency is also assumed to be vanishingly small, then the nature of the single boundary condition to be adopted in the limit $\sigma^{-1} \rightarrow 0$ depends, for the slow hydromagnetic wave, on the limiting value of $\sigma^{\frac{1}{2}}\Omega^2$. Similarly, if $\Omega \gg 1$, and the fast hydromagnetic wave is being considered, then the relevant boundary condition is found to depend on the limiting value of $\Omega\sigma^{-\frac{1}{2}}$.

The 'resistive' waves that are found to accompany the fast and slow waves, in order to satisfy the boundary conditions for small but finite values of σ^{-1}, are studied in some detail and their contribution to the wave damping is determined.

1. Introduction

In a previous paper published in this *Journal* (Woods 1962), a dispersion relation was derived for the propagation of hydromagnetic waves along a magnetic field lying in the axial direction of a cylindrical, partially ionized plasma. The neutral gas was assumed to possess viscosity and pressure, while the ionized gas in addition to these properties had a resistivity σ^{-1}. This complexity yielded a rather involved relationship between the axial wave-number k, the radial wave-number k_c and the frequency ω, particularly as the ion-cyclotron frequency ω_{ci} was also taken into account. The main effect of the neutral gas—which will not concern us in this paper—was to modify the Alfvén speed, $v_A = B_0(\mu\rho_0)^{-\frac{1}{2}}$, where B_0 is the steady axial magnetic field and ρ_0 is the density of the ionized gas, by replacing ρ_0 by ρ_0/s, where s is a complex number depending on the ion-neutral collision frequency and the ionization level (see (W 59)—equation (59) of Woods 1962). For the special case in which pressure and viscosity can be neglected, the case usually realized in hydromagnetic wave experiments (Jephcott & Stocker 1962), the dispersion relation reduces to (cf. (W 60))

$$[k^2 - k_A^2(1+i\delta k_t^2)]\,[k_t^2 - k_A^2(1+i\delta k_t^2)] = k^2 k_t^2\,\Omega^2, \tag{1}$$

where $k_A^2 \equiv \omega^2/(v_A^2 s)$, $k_t^2 \equiv k_c^2 + k^2$ is the total wave-number, $\delta \equiv (\mu\omega\sigma)^{-1}$ (mks units), and $\Omega \equiv \omega/\omega_{ci}$. For simplicity, in (1) we have not allowed for the anisotropic nature of the resistivity parameter δ, but it is easily verified from (W 60) that

$\delta_{\|}$ should replace δ appearing in the singular perturbation theory given in §2 of this paper.

With the magnetic field $\mathbf{B}$ expressed in cylindrical co-ordinates (r, θ, z) as

$$\mathbf{B}(r,\theta,z,t) = B_0\,\mathbf{n} + \mathbf{B}_1(r)\exp\{i(m\theta + kz - \omega t)\}, \tag{2}$$

where $\mathbf{n}$ is unit vector along the z-axis, the theory gave for the components of $\mathbf{B}_1$

$$\left.\begin{aligned} B_{1z} &= \sum_{s=1}^{2} k_{cs}\mathscr{A}_s J_m(k_{cs}r), \\ B_{1\theta} &= -\sum_{s=1}^{2} \{\mathscr{C}_s J'_m(k_{cs}r) + (km/k_{cs}r)\,\mathscr{A}_s J_m(k_{cs}r)\}, \\ B_{1r} &= i\sum_{s=1}^{2} \{k\mathscr{A}_s J'_m(k_{cs}r) + (m/k_{cs}r)\,\mathscr{C}_s J_m(k_{cs}r)\}, \end{aligned}\right\} \tag{3}$$

where
$$-\frac{\mathscr{C}_s}{\mathscr{A}_s} = \frac{k_{ts}^2 - k_A^2(1 + i\delta k_{ts}^2)}{k\Omega} = \frac{kk_{ts}^2\,\Omega}{k^2 - k_A^2(1+\delta i k_{ts}^2)} \quad (s = 1, 2), \tag{4}$$

$k_{ts}^2 = k_{cs}^2 + k^2$, and k_{t1}^2, k_{t2}^2 are the roots of the quadratic in k_t^2 in (1).

Suppose the plasma has a radius r_0; then the boundary conditions at $r = r_0$ depend on the conductivity of the material that encloses the plasma: for infinitely conducting walls

$$B_{1r} = 0 \quad \text{at} \quad r = r_0, \tag{5}$$

while for perfectly insulating walls, it is shown by Woods (1962) that continuity of the magnetic field leads to

$$\left.\begin{aligned} (m/r_0)\,B_{1z} - kB_{1\theta} &= 0 \quad (6a)\\ B_{1r} + iX_m B_{1z} &= 0 \quad (6b) \end{aligned}\right\} \quad \text{at} \quad r = r_0,$$

where
$$X_m \equiv K'_m(kr_0)/K_m(kr_0). \tag{7}$$

Equation (6a) is equivalent to requiring the radial current to vanish. The electric fields are assumed to be screened by an electric dipole layer on the walls (see discussion in earlier paper). The intermediate case of finite conductivity has also been considered.

Now for a given plasma, magnetic field strength B_0, and frequency ω, apart from the amplitude of the perturbations—which depends on the initial conditions —(3) contain five unknowns, viz. k, k_{c1}, k_{c2}, and the amplitude ratios,

$$a \equiv \mathscr{A}_2/\mathscr{A}_1, \quad c \equiv \mathscr{C}_2/\mathscr{C}_1. \tag{8}$$

One restriction on a/c follows from (4), while (1) imposes two further relations between k, k_{c1}, and k_{c2}; consequently two boundary conditions can be satisfied. If only one radial mode is present, say that corresponding to k_{c1}, then a similar argument shows that only one boundary condition can be satisfied. Thus by (5), if the walls are infinitely conducting, each of the two radial modes summed in (3) can be propagated separately,† whereas with insulating walls both modes must be present so that (6) can be satisfied.

Now suppose that the plasma resistivity is zero, i.e. $\delta = 0$; then (1) will yield only one root for k_t^2, and hence only one radial mode. Thus in this limiting case

† A remark that assumes that electric dipole layers are present, otherwise the modes are coupled by the vanishing of the tangential electric field. Whether or not such layers do occur on highly conducting walls is not quite clear; however, experimentally it does appear that the two modes in (1) can be propagated separately.

the two boundary conditions in (6) must be replaced by a single condition. This suggests that when δ is very small, but not quite zero, a boundary layer will form providing a rapid transition in one or more of the magnetic field components, and so permitting both of (6) to be satisfied at the wall. Then in the limit $\delta = 0$, the boundary layer collapses into a current sheet.

This point was not pursued in the earlier paper, because at that time most of the Culham Laboratory experiments (Jephcott & Stocker 1962), for which the theory was developed, involved waves with negligible B_{1r} and B_{1z} fields, so that it was necessary to satisfy only (6*a*). However, recent experiments on fast waves in insulating tubes involve relatively large B_{1r} and B_{1z} fields, and so require the analysis to be taken further for the special case when δ is small but finite. Of course the three relations between a, c, k, k_{c1}, and k_{c2}, mentioned above, together with (6), provide a means of finding the solution for the general case, but this would necessarily be a numerical rather than an analytic solution, and would throw no light on the limiting case described above.

2. The boundary condition for small plasma resistivity

On ignoring a term of order δ^2, (1) can be written†

$$i\delta k_A^2 k_t^4 + \{k^2\Omega^2 + k_A^2 - k^2 - i\delta k_A^2(2k_A^2 - k^2)\} k_t^2 + k_A^2(k^2 - k_A^2) = 0. \tag{9}$$

If ϵ is small $i\epsilon x^2 + bx + c = 0$ has approximate roots $-c/b$, ib/ϵ; so (9) has the approximate roots

$$k_{t1}^2 \approx \frac{(k_A^2 - k^2)\,k_A^2}{k^2\Omega^2 + k_A^2 - k^2 - i\delta k_A^2(2k_A^2 - k^2)} \approx \frac{k_A^2(k^2 - k_A^2)}{k^2 - k_A^2 - k^2\Omega^2}, \tag{10}$$

$$k_{t2}^2 \approx -i(k^2 - k_A^2 - k^2\Omega^2)/(\delta k_A^2). \tag{11}$$

For simplicity we shall neglect here the effect of the neutral gas and take k_A^2 to be real. Let η, ϵ, k_c and ζ denote real numbers defined by

$$k = \eta + i\epsilon, \quad k_{c1} = k_c - i\zeta. \tag{11a}$$

For the wave to be propagated a significant distance, we require that $\epsilon \ll \eta$; assuming this to be the case and neglecting second-order terms, we find from the first of (10) that

$$\eta^2 = k_A^2/h - \tfrac{1}{2}k_c^2 \pm G/2h, \tag{12}$$

$$\epsilon = (k_A^2\delta/2\eta h^2)\{(2-h)\,k_A^2 + \tfrac{1}{2}hk_c^2 \pm \tfrac{1}{2}G^{-1}[8k_A^4\Omega^2 + hk_c^2(hk_c^2 + 2\Omega^2k_A^2)]\} + (\zeta k_c/2\eta)\{1 \pm G^{-1}(2k_A^2 - hk_c^2)\}, \tag{13}$$

where $$h \equiv 1 - \Omega^2, \quad G \equiv (k_c^4h^2 + 4k_A^4\Omega^2)^{\frac{1}{2}}. \tag{14}$$

The positive sign in (12) gives a 'slow' wave and the negative sign a 'fast' wave.

To first order (11) becomes

$$k_{c2}^2 = -i\frac{\eta^2h - k_A^2}{\delta k_A^2} = \begin{cases} -2iA_s^2 & \text{(slow wave)} \\ 2iA_f^2 & \text{(fast wave)} \end{cases}, \tag{15}$$

where $$A_f \equiv \tfrac{1}{2}\{(hk_c^2 + G)/\delta k_A^2\}^{\frac{1}{2}}, \quad A_s \equiv \tfrac{1}{2}\{(G - hk_c^2)/\delta k_A^2\}^{\frac{1}{2}}. \tag{16}$$

† From (1) this approximation is equivalent to the restriction $\omega\rho/\sigma B_0^2 \ll 1$, which is satisfied in the experiments reported by Jephcott & Stocker (1962). However in an earlier paper by Klozenberg, McNammara & Thoneman (1963) the case $\omega\rho/\sigma B_0^2 \gg 1$ is investigated theoretically. This case is important for waves of much higher frequency than considered in this paper.

Hence
$$k_{c2} = \begin{cases} (1-i)\,A_s & \text{(slow wave)} \\ (1+i)\,A_f & \text{(fast wave)} \end{cases}, \tag{17}$$
where, as the plasma resistivity is assumed small, both A_s and A_f are large numbers. When Ω is small
$$A_f \approx (k_c/k_A)\,(2\delta)^{-\frac{1}{2}} \quad \text{and} \quad A_s \approx (k_A/k_c)\,\Omega(2\delta)^{-\frac{1}{2}}. \tag{18}$$

From the asymptotic form of the Bessel functions we have for the fast wave
$$J'_m(k_{c2}r) = J'_m\{(1+i)\,A_f r\} \sim -F_f(r), \tag{19}$$
where
$$F_f = \frac{1}{2}\left(\frac{2^{\frac{1}{2}}}{A_f \pi}\right)^{\frac{1}{2}} \exp\{i\pi(m+\tfrac{5}{4})/2\}\exp\{A_f(1-i)\,r\}\,r^{-\frac{1}{2}}, \tag{20}$$
provided, of course, that $r \neq 0$. Also $J_m(k_{c2}r) \sim iF_f(r)$. It now follows from (6), (7) and (8), that when δ is sufficiently small to make $A_f r_0$ large, the boundary conditions for insulating walls can be expressed in the form
$$\left.\begin{aligned} J'_m + \lambda_1\{(k^2+k_{c1}^2)/k_{c1}\}\,J_m &= F_f(r_0)\,\{c-\lambda_1 a A_f(1-i)\}, \\ kJ'_m + \{k_{c1}X_m + \lambda_2/k_{c1}\}\,J_m &= F_f(r_0)\,\{c\lambda_2/[A_f(1-i)] - X_m a A_f(1-i)\}, \end{aligned}\right\} \tag{21}$$
where $\lambda_1 \equiv (m/kr_0)\,(\mathscr{A}_1/\mathscr{C}_1)$, $\lambda_2 \equiv (m/r_0)\,(\mathscr{C}_1/\mathscr{A}_1)$, and the argument $k_{c1}r_0$ of the Bessel functions has been omitted for brevity. The corresponding boundary conditions for the slow wave are obtained by replacing $A_f(1+i)$ by $A_s(1-i)$ in (20) and (21), but notice from (18) that at low frequencies, for the asymptotic theory to be valid in this case, δ must be small enough to satisfy $(2\delta)^{\frac{1}{2}} \ll (k_A r_0\,\Omega/k_c)$.

From (4), (8) and (11)
$$a/c = (\mathscr{C}_1/\mathscr{A}_1)\,(k^2-k_A^2-i\delta k_{c2}^2k_A^2)/(kk_{c2}^2\,\Omega) = (\mathscr{C}_1/\mathscr{A}_1)\,k\Omega[A_f(1-i)]^{-2} \ll 1$$
and hence on eliminating F_f from (21) one finds
$$\left\{kJ'_m + \left(k_{c1}X_m + \frac{m}{r_0 k_{c1}}\frac{\mathscr{C}_1}{\mathscr{A}_1}\right)J_m\right\} \equiv k(1+i)\,l_f\left\{J'_m + \frac{m}{r_0 k}\frac{k^2+k_{c1}^2}{k_{c1}}\frac{\mathscr{A}_1}{\mathscr{C}_1}J_m\right\}, \tag{22}$$
where
$$l_f \equiv (\mathscr{C}_1/\mathscr{A}_1)\,(m/r_0k - X_m\,\Omega)/2A_f. \tag{23}$$
As A_f has been assumed large, $l_f \ll 1$ unless $(\mathscr{C}_1/\mathscr{A}_1)$ is large. From (4) and (12) we find that for the fast wave there are no critical frequencies or wave-numbers at which $(\mathscr{C}_1/\mathscr{A}_1)$ is unduly large, and so l_f is small, except perhaps at high frequencies. Thus in evaluating it we can neglect the small imaginary parts of k and k_{c1}. To first order then, by (4) and (11a) $\mathscr{C}_1/\mathscr{A}_1 = \Omega\eta(k_c^2+\eta^2)/(k_A^2-\eta^2)$, and using (12) to eliminate η^2 from this expression (take the negative sign in (12)) and then substituting the result in (23) we find
$$l_f = 2\eta\Omega k_A^3\delta^{\frac{1}{2}}(G+hk_c^2)^{-\frac{3}{2}}\left(\frac{m}{r_0\eta} - X_m\,\Omega\right), \tag{24}$$
where A_f has been eliminated by (16).

The corresponding boundary condition for the slow wave is like (22) except that $(1+i)\,l_f$ is replaced by $(1-i)\,l_s$, where
$$l_s = \tfrac{1}{4}(\eta/\Omega^2)\,k_A^{-3}\delta^{\frac{1}{2}}(G+hk_c^2)^{\frac{3}{2}}\,[m/(r_0\eta) - X_m\,\Omega]. \tag{25}$$
Notice that as $(G+hk_c^2) \to 2k_c^2$ when $\Omega \to 0$, and $k_A \equiv \omega/v_A$, l_f tends to zero like ω^4 at small frequencies, whereas l_s tends to infinity like ω^{-4}.

In (22) we now have one combined boundary condition from which $k_{c1} = k_c - i\zeta$ can be determined. When l_f is small, the dominant condition is the vanishing of the first term in curly brackets, which corresponds to (6*b*), whereas if l_f is large, as is possible at high frequencies, the vanishing of its bracketed coefficient is the dominant condition, corresponding to (6*a*). For the slow wave the dominant condition is (6*a*) at low frequencies and (6*b*) at high frequencies. The imaginary part of k_{c1}, i.e. ζ, is likely to be largest when l_f and l_s are near unity in value, and by (13) the wave attenuation is likely to be largest in this case.

3. Axisymmetric waves ($m = 0$)

If $m = 0$ the theory simplifies a little, and as this is an important case in experimental work, we shall consider it in some detail in this section. The analysis to be given below could easily be extended to the more general case of $m \neq 0$.

When $m = 0$ (11*a*) and (22) give

$$(\eta + i\epsilon)(1 - l_f - il_f) J_1[(k_c - i\zeta) r_0] - (k_c - i\zeta) X_0 J_0[(k_c - i\zeta) r_0] = 0, \quad (26)$$

where $X_0 = -K_1[(\eta + i\epsilon) r_0]/[(\eta + i\epsilon) r_0]$ and l_f is a function of k_c and η. Equation (26) provides two relations from which the values of k_c and ζ can be determined. If ϵ, ζ and l_f are assumed small it can be written

$$(\eta + i\epsilon)(1 - l_f - il_f)(J_1 - i\zeta r_0 J_1') + (k_c - i\zeta)(X_0 + i\epsilon r_0 X_0')(J_0 + i\zeta r_0 J_1) = 0, \quad (27)$$

where the argument of the J's is $k_c r_0$ and of X_0 is ηr_0. The real and imaginary parts of (27) yield

$$\frac{J_1(x)}{x J_0(x)} + \frac{f(y)}{1 - l_f(y)} = 0, \quad (28)$$

and

$$\zeta r_0\{1 + x/y + 1/f(y) + x^2 f(y)\} - x\{l_f(y) + (\epsilon/\eta)\, y f'(y)/f(y)\} = 0, \quad (29)$$

where

$$f(y) \equiv K_1(y)/\{y K_0(y)\}, \quad x \equiv k_c r_0, \quad y \equiv \eta r_0. \quad (30)$$

From (12) and (24)

$$l_f(y) = 2y^2 \Omega^2 b^2 f(y)(k_A^2 \delta)^{\frac{1}{2}} \{2[b^2 - hy^2]\}^{-\frac{3}{2}}, \quad (31)$$

where $b \equiv k_A r_0$. Equation (12) can be written

$$y^2 = b^2/h - \tfrac{1}{2}x^2 - (h^2 x^4 + 4b^4 \Omega^2)^{\frac{1}{2}}/2h, \quad (32)$$

and the problem is now reduced to that of solving (28) and (32) simultaneously for x and y; the solution will depend on the three non-dimensional numbers b, Ω, and $(k_A^2 \delta)^{\frac{1}{2}}$.

When x and y are found, (13) and (29) yield for the damping ratio

$$\frac{\epsilon}{\eta} = \frac{k_A^2 \delta}{2h^2 y^2 Q}\{(2 - h) b^2 + \tfrac{1}{2}hx^2 - \tfrac{1}{2}[8b^4\Omega^2 + hx^2(hx^2 + 2b^2\Omega^2)][h^2x^4 + 4b^4\Omega^2]^{-\frac{1}{2}}\} + (\Omega^2 - 1) x^2 l_f \{Q(2b^2 - 2hy^2 - hx^2)\}^{-1}, \quad (33)$$

where

$$Q \equiv 1 + f' hx^2 y(2b^2 - 2hy^2 - hx^2)^{-1}\{1 + x^2 f^2 + (1 + x/y) f\}^{-1}. \quad (34)$$

Notice that the additional damping due to the imaginary part of the radial wave-number, k_{c1}, is proportional to $(k_A^2 \delta)^{\frac{1}{2}}$ so that at small resistivity and large Alfvén wave velocity, v_A, this component would be the dominant term in (33) except near the critical frequencies $\Omega = 1$ and $\omega = k_c v_A$ (see discussion by Woods 1962 of the damping near these frequencies). The condition $\epsilon/\eta \ll 1$ imposed in §2 requires that the critical frequencies be avoided and that $k_A \delta^{\frac{1}{2}} \ll 1$, i.e. that $(\omega\rho/\sigma B_0^2)^{\frac{1}{2}} \ll 1$, see footnote on p. 403.

Now consider the slow wave under such conditions that the parameter l_s is *large*. By (25) and (30),

$$l_s = \tfrac{1}{4}(\Omega b^4)^{-1} y^2 f(y)\,(k_A^2\delta)^{\frac{1}{2}}\{r_0^2 G + hx^2\}^{\frac{3}{2}}. \tag{35}$$

By the same type of analysis as that used for the fast wave, corresponding to (28), (29) and (32) we find

$$\frac{J_1(x)}{xJ_0(x)} - \frac{f(y)}{2l_s(y)} = 0, \tag{36}$$

$$\zeta r_0 = xf(y)/2l_s, \tag{37}$$

and

$$y^2 = b^2/h - \tfrac{1}{2}x^2 + (h^2x^4 + 4b^4\Omega^2)^{\frac{1}{2}}/2h. \tag{38}$$

From (13) and (37), the additional damping, ϵ^* say, due to ζ is

$$\frac{\epsilon^*}{\eta} = \frac{x^2}{2y^2 l_s} f(y)\,\{1 + (2k_A^2 - hk_c^2)/G\} \tag{39}$$

and at small values of Ω (13) and (39) yield

$$\epsilon/\eta = k_A^2\delta(k_A^2 + k_c^2)/2\eta^2 + 2\Omega k_A^6/r_0\eta^4 k_c^3 k_A \delta^{\frac{1}{2}}. \tag{40}$$

The condition $\epsilon/\eta \ll 1$ is met if $(\Omega k_A^2/k_c^2) \ll k_A\delta^{\frac{1}{2}} \ll 1$, and the first of these inequalities is a consequence of $l_s \gg 1$. For the case, $l_s \ll 1$, which corresponds to Ω remaining finite and δ tending to zero, we need only replace l_f in (28) and (29) by l_s and $-l_s$, respectively.

4. The effects of electron inertia

The singular perturbation problem studied in this paper arises from the fact that putting resistivity equal to zero lowers the order of the differential equation. That this is a consequence of being able to neglect electron inertia and hence to admit the possibility of current sheets, in the theory of hydromagnetic waves, can be shown as follows. Let $\mathbf{j}$ be the current and n the number density of electrons; then inclusion of electron inertia adds a term $(m_e/e^2n)\,\partial\mathbf{j}/\partial t$ to $\mathbf{j}/\sigma$ in Ohm's law. Thus for oscillatory perturbations the coefficient of $\mathbf{j}$ is changed from σ^{-1} to $\sigma^{-1} - i(\omega m_e/e^2n)$, and in place of δ there is now

$$\delta = (\sigma\mu\omega)^{-1} - i(c/\omega_{pe})^2, \tag{41}$$

where c is the speed of e.m. waves and ω_{pc} is the electron plasma frequency.

In the limit $\sigma \to \infty$, $\delta \to -i(c/\omega_{pe})^2$, and as a consequence $k_{c2} \to R_1(\omega_{qe}/c)$, and $(1+i)\,l_f \to R_2(c/\omega_{pe})$, where R_1 and R_2 are real numbers. Thus the limit $\delta = 0$ is not theoretically attainable, although for hydromagnetic waves electron inertia is negligible and the limit is practically attainable.

5. Expressions for the magnetic field strength

From (3), (8), (20) and (21) we find that for $m = 0$, the magnetic field perturbations are given by

$$\left.\begin{aligned} B_{1z} &= \mathscr{A}_1\{k_{c1}J_0(k_{c1}r) - \mathscr{L}T(r)\},\\ B_{1\theta} &= -\mathscr{C}_1\{J_0'(k_{c1}r) - J_0'(k_{c1}r_0)\,T(r)\},\\ B_{1r} &= ik\mathscr{A}_1\{J_0'(k_{c1}r) - (a/c)\,J_0'(k_{c1}r_0)\,T(r)\}\\ &\approx ik\mathscr{A}_1 J_0'(k_{c1}r), \end{aligned}\right\} \tag{42}$$

where
$$\mathscr{L} \equiv \{k/X_0(kr_0)\}\, J_0'(k_{c1}r_0) + k_{c1} J_0(k_{c1}r_0)$$
$$= \{k/X_0(kr_0)\}\,(1+i)\, l_f J_0'(k_{c1}r_0), \text{ using (26)} \tag{43}$$
and
$$T(r) \equiv \exp\{-(1-i)\, A_f r_0(1-r/r_0)\}\,(r_0/r)^{\frac{1}{2}}.$$

For the slow wave replace $A_f(1+i)$ and $l_f(1+i)$ by $A_s(1-i)$ and $l_s(1-i)$. The additional wave appearing in B_{1z} and $B_{1\theta}$ can be appropriately termed the 'resistive' wave; its effect on B_{1r} is negligible, while its effect on B_{1z} and $B_{1\theta}$ is only important near the boundary.

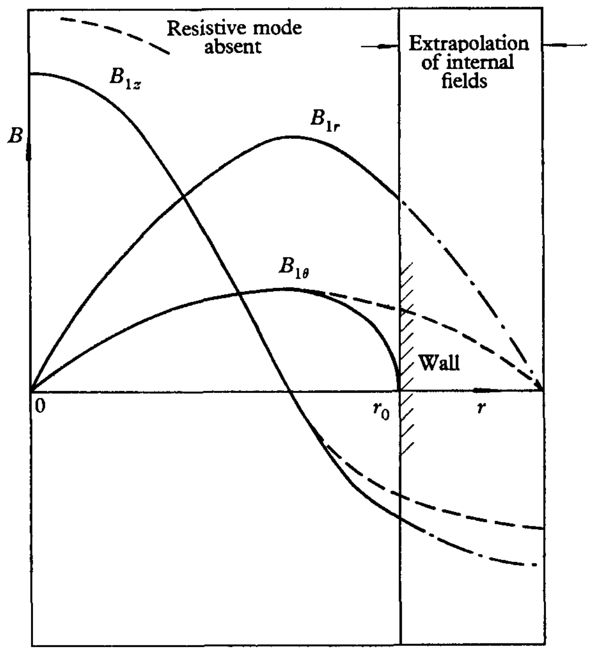

FIGURE 1. Fast wave, $l_f \ll 1$.

(i) *Fast wave*: $l_f \ll 1$

For this case $\mathscr{L}$ is small and so is the contribution of the resistive mode to B_{1z}. As $T(r_0) = 1$, $B_{1\theta} = 0$ at $r = r_0$, no matter how small the resistivity is, and the effect of the resistive mode on $B_{1\theta}$ is quite large as shown in figure 1.

The axial current is (see (W 38)) $\mu j_{1z} = \mathscr{C}_1\{kJ_0(k_{c1}r) + (1-i)\, A_f J_0'(k_{c1}r)\, T(r)\}$, which becomes very large at $r = r_0$, matching the rapid fall in $B_{1\theta}$. Thus the insulating wall appears to behave like an infinitely conducting wall, by producing large axial currents in the plasma adjacent to it. However, from (3) and (5) it follows that for an infinitely conducting wall B_{1r}, and as a consequence $B_{1\theta}$ (recall that $m = 0$ in the present discussion) tends to zero *smoothly* at $r = r_0$, while B_{1z} has a discontinuity, corresponding to an azimuthal current sheet, at $r = r_0$. The cases are therefore quite different.

(ii) *Slow wave*: $l_s \gg 1$

In this case, as $\mathscr{L}$ remains finite it follows from (43) that $J_0'(k_{c1}r_0) \approx 0$, and so $\mathscr{L} \approx k_{c1}J_0(k_{c1}r_0)$, and
$$B_{1z} \approx \mathscr{A}_1 k_{c1}\{J_0(k_{c1}r) - J_0(k_{c1}r_0)\, T(r)\}, \quad B_{1\theta} \approx \mathscr{C}_1 J_0'(k_{c1}r).$$

The situation is now that the resistive mode has a big effect on B_{1z} near the wall, but little effect on $B_{1\theta}$ and B_{1r} (figure 2). It is this case that corresponds closely to the conducting wall case—paradoxically, the case that arises when the resistivity is not zero, but is large enough to satisfy the inequalities

$$\tfrac{1}{2}f(k_A r_0)\,(r_0 k_c)^2 \gg r_0(k_A/k_c)\,\Omega(2\delta)^{-\frac{1}{2}} \gg 1, \tag{44}$$

which follow from (18), (35) and $A_s r_0 \gg 1$, $l_s \gg 1$, $\Omega \ll 1$.

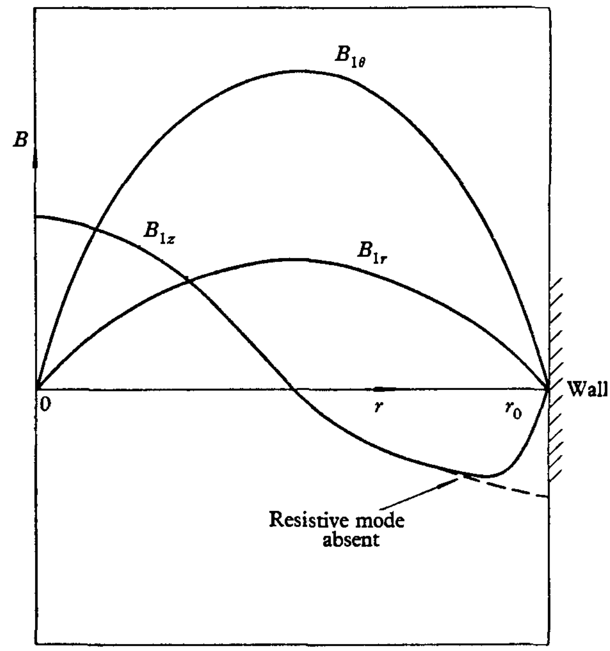

FIGURE 2. Slow wave, $l_s \gg 1$.

The magnetic field distributions for the two other cases, viz. (iii) fast wave, $l_f \gg 1$ and (iv) slow wave, $l_s \ll 1$, will resemble (ii) and (i), respectively, except that the relative amplitudes will not be as shown in the figures. In (iii) B_{1z}, $B_{1r} \gg B_{1\theta}$, while in (iv) $B_{1\theta} \gg B_{1z}$, B_{1r}, except near the critical frequency $\Omega = 1$.

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